THE ERGODIC INFINITE MEASURE PRESERVING TRANSFORMATION OF BOOLE

BY

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ABSTRACT

G. Boole proved that the transformation φ of the real line, defined by $\varphi(x) = x - 1/x$, preserves Lebesgue measure. A general method is applied to proving that φ is ergodic. Some further applications of the method are also indicated.

1. Introduction

It was over a hundred years ago that G. Boole [2: 780] discovered the surprising formula

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x - (1/x))dx, \tag{1}$$

which is valid for any integrable function f. In the ensuing years at least one author generalized and extended (1), (see [3], [4]) and used it to derive various definite integrals. An ergodic-theorist, upon encountering (1) will immediately recognize that what is being expressed is the fact that the transformation $x \to x - (1/x)$ preserves the usual Lebesgue measure on the real line $\mathbb{R}$. Now as is well known there are fundamental differences between the measure preserving transformations of finite measure spaces and those of infinite measure-spaces. In particular the latter theory suffers from a paucity of good examples, and so a natural question arose — what can ergodic theory say about Boole's transformation $\varphi(x) = x - (1/x)$. We embark upon such a study in this paper which is devoted in the main to showing φ is ergodic.

The method of proof that will be used has wider applicability. In particular it can be used to show that φ is totally ergodic, i.e., φ^k is ergodic for all $k \geqq 1$.

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It also applies to certain transformations of $\mathbb{R}$ that do not preserve Lebesgue measure, by first constructing for them an invariant measure and then proving ergodicity. It also applies to the study of the questions raised in [1]. For instance Example 2 of [1] is an isomorphic image of Boole's transformation under the change of variable $x \to \arg((x-i)/(x+i))$. The mixing properties of φ are probably not exhausted by the statement that φ is totally ergodic, and the proof to be given here suggests that more is true. We hope to return to some of these points in the future.

A brief word about the organization of the paper: Section 2 contains some general information about measure-preserving transformations of infinite measure spaces. Section 3 contains the main theorem, namely that φ is ergodic, along with several lemmas necessary for its proof.

Finally we wish to acknowledge an astute observation by Professor L. Flatto which somewhat simplified our original version of this work. He pointed out how to replace asymptotic analysis of the sequences $\{x_n\}$ and $\{u_n\}$ by the integral test in Lemma 3.2.

2. Induced transformations

Let $(X, \mathbb{B}, \mu)$ be a sigma-finite measure space. Let φ be a measure-preserving transformation of X onto X which is not necessarily invertible; that is $\varphi^{-1}(E) \in \mathbb{B}$ and $\mu(\varphi^{-1}E) = \mu(E)$ for every $E \in \mathbb{B}$. Let A be a measurable subset of X such that

$$X = \bigcup_{n=1}^{\infty} \varphi^{-n}A. \tag{1}$$

For each $x \in X$ we define a positive integer $\tau(x)$ by $\tau(x) \equiv \inf\{n \mid n \geqq 1, \varphi^n x \in A\}$. The sets $\{x \mid \tau(x) = n\}$, $n = 1, 2, \cdots$, form a disjoint partition of X and satisfy

$$\{x \mid \tau(x) = n+1\} = \varphi^{-1}\{x \mid \tau(x) = n\} \cap \varphi^{-1}A^c \tag{2}$$

for $n = 1, 2, \cdots$. Let us define the following subsets of $\varphi^{-n}A$:

$$A_n \equiv \{x \mid \tau(x) = n\} \cap A \text{ and } B_n \equiv \{x \mid \tau(x) = n\} \cap A^c, \qquad n = 1, 2, \cdots.$$

For these sets we have

$$\varphi^{-1}B_n = B_{n+1} \cup A_{n+1} \text{ (disjoint)}, \tag{3}$$

$n = 1, 2, \cdots$. It is helpful to form a mental image of the mapping in terms of a two-story building† (see Fig. 1).

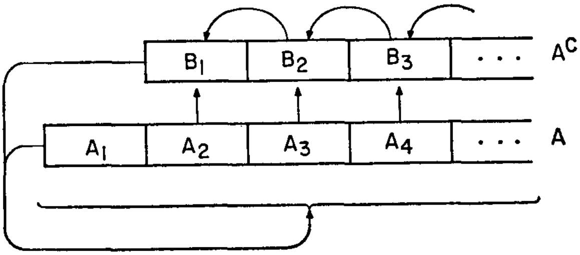

Fig. 1

The transformation φ_A mapping A onto A defined by $\varphi_A x \equiv \varphi^{\tau(x)} x$ for $x \in A$ is called *the induced transformation* by φ on A. It is easy to see that

$$\varphi_A^{-1} E = \bigcup_{n=1}^{\infty} A_n \cap \varphi^{-n} E \text{ (disjoint)} \tag{4}$$

for $E \subset A$, $E \in \mathbb{B}$.

THEOREM 1. *If $\mu(A) < \infty$ then φ_A preserves the measure μ on $\mathbb{B}_A$, the measurable subsets of A.*

PROOF. Using (3) and the fact that φ is μ-measure preserving we can show by induction

$$\mu(E) = \sum_{k=1}^{n} \mu(A_k \cap \varphi^{-k} E) + \mu(B_n \cap \varphi^{-n} E) \tag{5}$$

for $E \subset A$, $E \in \mathbb{B}$, and $n = 1, 2, \cdots$. By the same method

$$\mu(B_n) = \sum_{k=n+1}^{\infty} \mu(A_k) \tag{6}$$

$n = 1, 2, \cdots$. From (4) and (5) we have

$$\left| \mu(E) - \mu(\varphi_A^{-1} E) \right| = \sum_{k=n+1}^{\infty} \mu(A_k \cap \varphi^{-k} E) + \mu(B_n \cap \varphi^{-n} E) \tag{7}$$

for all n. From (6) the right-hand side of (7) is dominated by $2 \sum_{k=n+1}^{\infty} \mu(A_k)$. This quantity is the tail of a convergent series because $\mu(A) < \infty$; hence it converges to zero as $n \to \infty$ forcing $\mu(E) = \mu(\varphi_A^{-1} E)$ for $E \subset A$, $E \in \mathbb{B}$.

† In the case of invertible transformations the usual picture is somewhat different [5: 29].

THEOREM 2. *If φ_A is ergodic then so is φ.*

PROOF. Let E be an invariant measurable set of positive measure. Invariant here means $\varphi^{-1}E = E$. At least one of the sets $A \cap E$ and $A \cap E^c$ has positive measure. Since the roles of E and E^c are interchangeable in this argument let us assume $\mu(A \cap E) > 0$. From (4)

$$\varphi_A^{-1}(E \cap A) = \bigcup_{n=1}^{\infty} A_n \cap \varphi^{-n}(A \cap E) = \bigcup_{n=1}^{\infty} A_n \cap \varphi^{-n}A \cap E$$

$$= \bigcup_{n=1}^{\infty} A_n \cap E = E \cap A.$$

Since φ_A^{-1} is ergodic, we then have $E \cap A = A$ modulo a set of measure zero. Thus $X = \bigcup_{n=1}^{\infty} \varphi^{-n}(E \cap A) = \bigcup_{n=1}^{\infty} E \cap \varphi^{-n}A = E$ modulo a set of measure zero.

REMARK. The converse of Theorem 2 is also true.

PROOF. Let φ be ergodic and E a φ_A-invariant subset of A of positive measure. Thus $\varphi_A^{-1}E = \bigcup_{n=1}^{\infty} A_n \cap \varphi^{-n}E = E$. Let $F = E \cup \bigcup_{n=1}^{\infty} B_n \cap \varphi^{-n}E$; consequently $\varphi^{-1}F = \varphi^{-1}E \cup \bigcup_{n=1}^{\infty} \varphi^{-1}B_n \cap \varphi^{-n}E$. From (3) and the fact that $\varphi^{-1}E = (A_1 \cup B_1) \cap \varphi^{-1}E$ we can conclude $\varphi^{-1}F = F$. Thus $F = X$ modulo a set of measure zero. Since $\bigcup_{n=1}^{\infty} B_n \cap \varphi^{-n}E \subset A^c$ we have $E \supseteq A$ modulo a set of measure zero.

3. Boole's transformation

Let us now turn our attention to the transformation $\varphi: x \to x - (1/x)$ which maps $\mathbb{R} - \{0\}$ onto $\mathbb{R}$ in a two-to-one manner.

To form the space X on which φ acts in accordance with Section 2 we must delete from $\mathbb{R}$ the denumerable set $\bigcup_{n=0}^{\infty} \varphi^{-n}\{0\}$ which consists of all orbits that hit 0, a point where φ is undefined. We shall adopt the notation $|E|$ for the Lebesgue measure of a measurable subset E of $\mathbb{R}$.

THEOREM. *φ preserves Lebesgue measure.*

PROOF. We shall reproduce a version of a proof which is to be found in an editor's note at the end of [3] (the editor was Cayley). We must show that $|\varphi^{-1}I| = |I|$ for arbitrary intervals I. It suffices to verify this for intervals of the form $I = (0,\eta)$, $\eta > 0$ and $I = (\eta, 0)$, $n < 0$. For $\eta > 0$ (see Fig. 2) we have

$$\varphi^{-1}(0,\eta) = (-1,\xi_1) \cup (1,\xi_2) \qquad \text{(disjoint)} \tag{1}$$

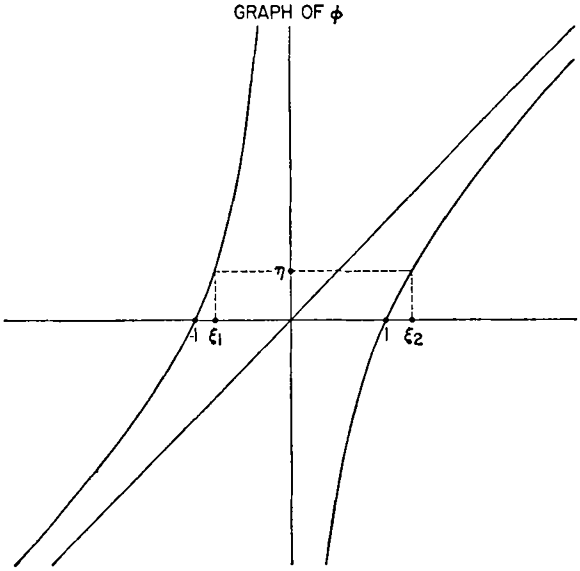

Fig. 2

where ξ_1 and ξ_2 are the negative and positive roots of

$$\eta = x - \frac{1}{x},$$

i.e.,

$$x^2 - \eta x - 1 = 0. \tag{2}$$

Since η equals the sum of the roots of (2) we have from (1)

$$\begin{aligned} |\varphi^{-1}(0,\eta)| &= \xi_1 + \xi_2 = \eta \\ &= |(0,\eta)|. \end{aligned}$$

The case $\eta < 0$ is handled similarly.

MAIN THEOREM. *φ is ergodic.*

The plan of the proof is to reduce the problem to the more familiar subject of ergodic theory on finite measure spaces. Therefore we intend to demonstrate that for the set $A = (-1, 1) \cap X$ which has finite measure the induced transformation φ_A is ergodic. Then Theorem 2 can be invoked to conclude the main theorem. In order to establish the ergodicity of φ_A we shall seek a property like condition (c) of Rényi [6], that is a uniform bound for a certain ratio of maximum to minimum derivative of φ_A^n.

This property bounds the departure from linearity of the higher iterates of φ_A over small intervals where φ_A^n is one-to-one. If φ_A^n were linear over these intervals the aforementioned ratio would always be equal to one and ergodicity would be easy to prove. However it is interesting to note that here is an example where even the theorem of [1] is useless and a more delicate analysis must be made.

First the formula for the derivative.

$$\varphi'(x) = 1 + (1/x^2) \tag{3}$$

is important. Next we require finer details to Fig. 1 for the specific case of Boole's transformation.

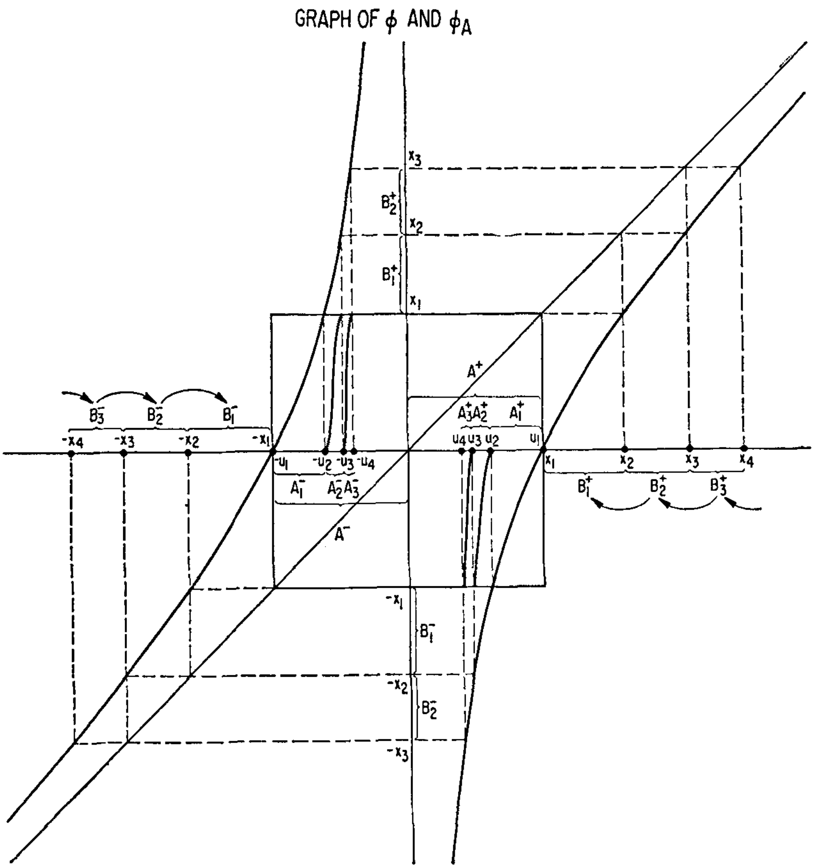

Fig. 3

For this we utilize the sequence x_n, $n = 1, 2, \cdots$ of strictly positive real numbers which are defined as follows. For $n = 1, 2, \cdots$ let x_{n+1} be chosen as the positive root of

$$x^2 - x_n x - 1 = 0$$

with the initial condition $x_1 = 1$. Note that

$$\begin{aligned} x_n &= \varphi(x_{n+1}) \\ &= x_{n+1} - (1/x_{n+1}) \end{aligned} \tag{4}$$

for $n = 1, 2, \cdots$. Since $\varphi'(x) > 0$, the sequence x_n, $n = 1, 2, \cdots$ is strictly increasing; and since φ has no fixed points among the finite real numbers, $\lim_{n\to\infty} x_n = \infty$. From the fact that φ is an odd function

$$-x_n = \varphi(-x_{n+1}), \tag{5}$$

$n = 1, 2, \cdots$; and it is easy to see that

$$(-x_n, x_n) \cap X = \varphi^{-1+n} A. \tag{6}$$

Consequently condition (2A.1) is satisfied and φ_A can be defined (see Fig. 3 for the graph of φ_A). The conditions of Theorem 1 are also satisfied which implies that φ_A preserves Lebesgue measure.

We need to define another sequence u_n, $n = 1, 2, \cdots$ where $u_1 = x_1 = 1$ and u_{n+1} is taken to be the positive root of the equation

$$x^2 + x_n x - 1 = 0$$

for $n = 1, 2, \cdots$. Note that

$$-x_n = \varphi(u_{n+1}) = u_{n+1} - \frac{1}{u_{n+1}}, \tag{7}$$

$n = 1, 2, \cdots$. It follows that

$$u_n - (1/u_n) + x_n - (1/x_n) = 0$$

$n = 1, 2, \cdots$, and

$$u_n + x_n = \frac{u_n + x_n}{u_n x_n};$$

hence

$$u_n x_n = 1. \tag{8}$$

Consequently $\lim_{n\to\infty} u_n = 0$. In addition we have

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1. \tag{9}$$

Fig. 3 depicts the needed additional information to Fig. 1. For instance it can be seen that

$$A_n = [(-u_n, -u_{n+1}) \cup (u_{n+1}, u_n)] \cap X$$
$$B_n = [(-x_{n+1}, -x_n) \cup (x_n, x_{n+1})] \cap X$$

$n = 1,2,\cdots$. Furthermore if we define the following sets

$$A_n^+ \equiv (u_{n+1}, u_n) \cap X$$
$$A_n^- \equiv (-u_n, -u_{n+1}) \cap X$$
$$B_n^+ \equiv (x_n, x_{n+1}) \cap X$$
$$B_n^- \equiv (-x_{n+1}, -x_n) \cap X$$

$n = 1,2,\cdots$ and

$$A^+ \equiv (0,1) \cap X$$
$$A^- \equiv (-1,0) \cap X,$$

we then have that φ is a one-to-one mapping of these sets onto others according to the manner listed below:

$$A_{n+1}^+ \to B_n^-$$
$$A_{n+1}^- \to B_n^+$$
$$B_{n+1}^+ \to B_n^+$$
$$B_{n+1}^- \to B_n^-$$

$n = 1,2,\cdots$, and

$$A_1^+ \to A^-$$
$$A_1^- \to A^+$$
$$B_1^+ \to A^+$$
$$B_1^- \to A^-.$$

Since $\varphi_A \,|\, A_n^{\pm} = \varphi^n$, the transformation φ_A is a one-to-one mapping of A_n^+ onto A^- and A_n^- onto A^+.

Next let α denote the partition of A into the sets A_n^+ and A_n^-, $n = 1,2,\cdots$.

Let $\alpha^{(n)}$ denote the common refinement $\alpha \vee \varphi_A^{-1}\alpha \vee \cdots \vee \varphi_A^{-n+1}\alpha$. The elements of $\alpha^{(n)}$ are of the form $I_1 \cap \varphi_A^{-1} I_2 \cap \cdots \cap \varphi_A^{-(n+1)} I_n$ where $I_1, \cdots, I_n \in \alpha$. It is also convenient to define $\alpha^{(0)}$ as the partition of A into sets A^+ and A^-. Let us denote a generic element of $\alpha^{(n)}$ by $I^{(n)}$ for $n = 0, 1, 2, \cdots$. Sometimes the superscript will be omitted for $n = 1$. We have that φ_A maps an $I^{(n)}$ one-to-one onto an $I^{(n-1)}$, $n = 1, 2, \cdots$ and $\varphi_A^{-1} I^{(0)} = \varphi_A I^{(0)} = A - I^{(0)}$.

In addition we can write $I^{(n)} = I^{(1)} \cap \varphi_A^{-1} I^{(n-1)}$. The closure $\overline{I^{(n)}}$ of an $I^{(n)}$ is a little interval whose size is governed by the following considerations. The mean value theorem yields

$$|I^{(n)}| = (\varphi_A'(x))^{-1} |I^{(n-1)}|$$

for some $x \in \overline{I^{(n)}}$. As indicated above the $I^{(n)}$ is a subset of some $I^{(1)}$ equal to one of the sets A_m^+ of A_m^- for an $m \geqq 1$. By the chain rule

$$\varphi_A'(x) = \varphi^{m\prime}(x) = \varphi'(x) \cdot \varphi'(\varphi x) \cdots \cdot \varphi'(\varphi^{m-1} x).$$

From (3) we have that $\varphi'(x) \geqq 1$ for $x \in \mathbb{R} - \{0\}$ and $\varphi'(x) \geqq 2$ for $x \in A$; hence $\varphi_A'(x) \geqq 2$ so that we get

$$|I^{(n)}| \leqq (1/2) I^{(n-1)}. \tag{10}$$

Applying this inequality successively we have

$$|I^{(n)}| \leqq 2^{-n}. \tag{11}$$

Actually we shall need a similar estimate for the ratio $|I^{(n)}|/|I^{(1)}|$ where $I^{(n)} = I^{(1)} \cap \varphi_A^{-1} I^{(n-1)}$, but this is best done later. At any rate from (11) follows the fact that α is a generator (one-sided); i.e., the smallest σ-field containing $\bigcup_{n=1}^{\infty} \alpha^{(n)}$ is the σ-field $\mathbb{B}_A$ of measurable subsets of A.

LEMMA 1. *There exists a constant C' such that for all $n \geqq 1$*

$$\frac{|\varphi^n x - \varphi^n y|}{|\varphi^n I|} \leqq C' \frac{|x - y|}{|I|} \tag{12}$$

whenever $x, y \in I$, $I = A_m^+$ or A_m^- and $n \leqq m$.

PROOF. Due to symmetry it suffices to prove (12) merely for $I = A_m^-$ in which case Fig. 4 applies.

The following string of equalities and inequalities are consequences of the mean value theorem, the chain rule, and (3).

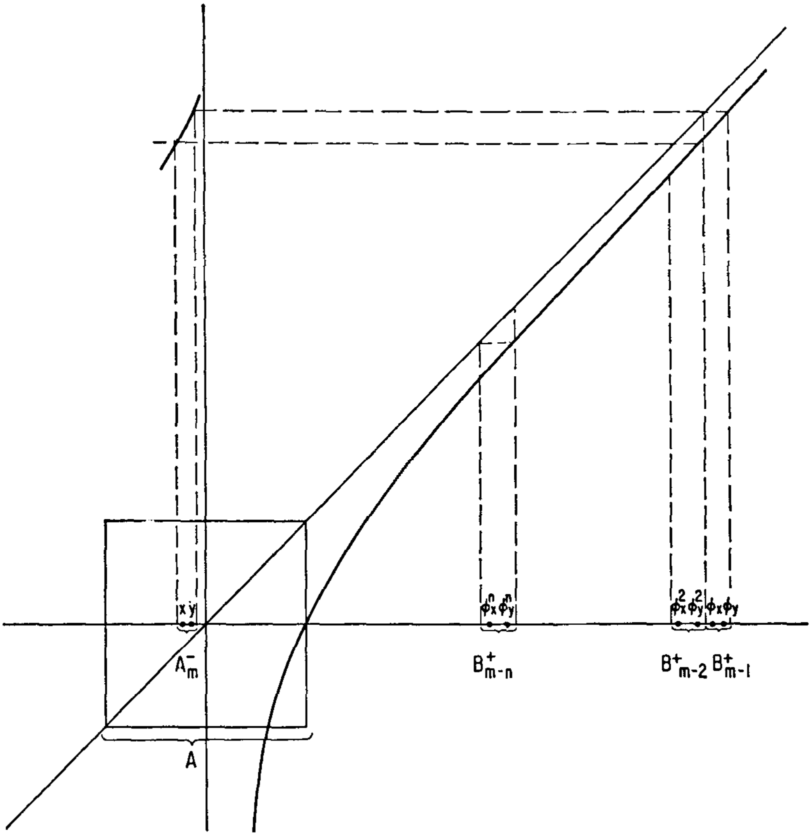

Fig. 4

For some $s, t, \in \overline{\varphi I}$ and $u, v \in \overline{A_m^-}$,

$$\frac{|\varphi^n x - \varphi^n y|}{|\varphi^n I|} = \frac{\varphi^{(n-1)\prime}(s)}{\varphi^{(n-1)\prime}(t)} \cdot \frac{\varphi'(u)}{\varphi'(v)} \cdot \frac{|x-y|}{|I|}$$

$$= \frac{\varphi'(s)\cdot\varphi'(\varphi s)\cdots\varphi'(\varphi^{n-2}s)}{\varphi'(t)\cdot\varphi'(\varphi t)\cdots\varphi'(\varphi^{n-2}t)} \quad \frac{\varphi'(u)}{\varphi'(v)} \cdot \frac{|x-y|}{|I|}$$

$$\leqq \frac{\varphi'(x_{m-1})\cdot\varphi'(x_{m-2})\cdots\varphi'(x_{m-n+1})}{\varphi'(x_m)\cdot\varphi'(x_{m-1})\cdots\varphi'(x_{m-n+2})} \cdot \frac{\varphi'(-u_{m+1})}{\varphi'(-u_m)} \cdot \frac{|x-y|}{|I|}$$

$$= \frac{\varphi'(x_{m-n+1})}{\varphi'(x_m)} \cdot \frac{\varphi'(-u_{m+1})}{\varphi'(-u_m)} \cdot \frac{|x-y|}{|I|}$$

$$\leqq 2 \cdot \frac{(1+u_{m+1}^{-2})}{(1+u_m^{-2})} \cdot \frac{|x-y|}{|I|}.$$

The desired

$$C' = \sup_m \left[\frac{2^{(1+u_{m+1}^{-2})}}{(1+u_m^{-2})} \right],$$

which is guaranteed to be finite by (9).

LEMMA 2. *There exist a constant C'' such that for all I*

$$\frac{\varphi_A'(x)}{\varphi_A'(y)} \leqq 1 + C'' \frac{|x-y|}{|I|} \tag{13}$$

whenever $x, y \in I$.

PROOF. Again we prove (13) merely for $I = A_n^-$, the case $I = A_n^+$ being entirely symmetrical. For $x, y \in A_n^-$ and $1 \leqq j \leqq n-1$ we have by the mean value theorem that there exists $z \in \overline{\varphi^j A_n^-} = \overline{B_{n-j}^+}$ such that

$$\varphi'(\varphi^j x) = \varphi'(\varphi^j y) + \varphi'(z)(\varphi^j x - \varphi^j y).$$

Thus

$$\frac{\varphi'(\varphi^j x)}{\varphi'(\varphi^j y)} = 1 + \frac{\varphi''(z)}{\varphi'(\varphi^j y} \cdot (\varphi^j x - \varphi^j y)$$

$$\leqq 1 + \frac{|\varphi''(z)|}{2} |\varphi^j x - \varphi^j y|,$$

by Lemma 1

$$\leqq 1 + C' \cdot \frac{|B_{n-j}^+|}{|z^3|} \cdot \frac{|x-y|}{|I|},$$

hence

$$\frac{\varphi'(\varphi^j x)}{\varphi'(\varphi^j y)} \leqq 1 + C' \cdot \frac{(x_{n-j+1} - x_{n-j})}{x_{n-j}^3} \cdot \frac{|x-y|}{|I|} \tag{14}$$

for $x, y \in I = A_n^-$ and $1 \leqq j \leqq n-1$.

Next by the mean value theorem again there exists $x \in \bar{I} = \overline{A_n^-}$ such that

$$\frac{\varphi'(x)}{\varphi'(y)} \leqq 1 + \frac{|\varphi''(z)|}{\varphi'(y)} |x-y|$$

$$= 1 + \frac{2}{|z^3|} \cdot \frac{|I|}{(1+(1/y^2))} \cdot \frac{|x-y|}{|I|}$$

$$\leqq 1 + \frac{2(u_n - u_{n+1})}{u_{n+1}^3(1 + (1/u_n^2))} \cdot \frac{|x - y|}{|I|}$$

$$= 1 + \frac{2u_n u_{n+1}^2}{u_{n+1}^3(1 + x_n^2)} \cdot \frac{|x - y|}{|I|}.$$

Thus

$$\frac{\varphi'(x)}{\varphi'(y)} \leqq 1 + O(1) \frac{|x - y|}{|I|} \tag{15}$$

for $x, y \in I$.

Finally applying the chain rule we have

$$\frac{\varphi_A'(x)}{\varphi_A'(y)} = \frac{\varphi^{n'}(x)}{\varphi^{n'}(y)} = \frac{\varphi'(\varphi^{n-1}x) \cdots \varphi'(\varphi x)}{\varphi'(\varphi^{n-1}y) \cdots \varphi'(\varphi y)} \cdot \frac{\varphi'(x)}{\varphi'(y)}. \tag{16}$$

From (14) and (15)

$$\frac{\varphi_A'(x)}{\varphi_A'(y)} \leqq \prod_{m=1}^{n-1} \left(1 + C' \frac{(x_{m+1} - x_m)}{x_m^3} \frac{|x - y|}{|I|}\right)\left(1 + O(1) \frac{|x - y|}{|I|}\right) \tag{17}$$

which by (9)

$$\leqq \prod_{m=1}^{\infty} \left(1 + O(1) \frac{(x_{m+1} - x_m)}{x_{m+1}^3} \frac{|x - y|}{|I|}\right)\left((1 + O(1) \frac{|x - y|}{|I|}\right).$$

Applying the integral test, $\int_1^\infty x^{-3}\,dx < \infty$, we have that the infinite series $\sum_{n=1}^\infty x_{m+1}^{-3}(x_{m+1} - x_m)$ converges.

Relation (13) then follows from the following elementary exercise.

PROPOSITION. *If $\{a_n\}$ is a sequence of nonnegative numbers such that $\sum_{n=1}^\infty a_n < \infty$ then*

$$\prod_{n=1}^{\infty} (1 + xa_n) = 1 + O(x)$$

for $0 \leqq x \leqq 1$.

PROOF.

$$1 + x \sum_{n=1}^{\infty} a_n \leqq \prod_{n=1}^{\infty} (1 + xa_n) \leqq e^x \sum_{n=1}^{\infty} a_n$$

$$\leqq 1 + x \sum_{n=1}^{\infty} a_n + \left(x \sum_{n=1}^{\infty} a_n\right)^2 \Big/ 2! + \cdots.$$

$$\leqq 1 + x \exp\left(\sum_{n=1}^{\infty} a_n\right) \quad \text{for } 0 \leqq x \leqq 1.$$

We now define the important quantities for the study of the present type of transformation.

DEFINITION. Let

$$M_n(I^{(n)}) \equiv \sup_{x,y \in I^{(n)}} \frac{\varphi_A^{n\prime}(x)}{\varphi_A^{n\prime}(y)}$$

and

$$M_n \equiv \sup_{I^{(n)} \in \alpha^{(n)}} M_n(I^{(n)})$$

for $n = 1, 2, \cdots$.

LEMMA 3. *If* $E \subseteq I^{(0)}$, $|E| > 0$ *and* $I^{(n)} \subset \varphi_A^{-n} I^{(0)}$, *then*

$$1/M_n(I^{(n)}) \leqq \frac{|I^{(n)} \cap \varphi_A^{-n} E|}{|I^{(n)}||E|} \leqq M_n(I^{(n)}). \tag{18}$$

PROOF. Since φ_A^n is a differentiable map of $\overline{I^{(n)}}$ onto $\overline{I^{(0)}}$ we can apply the mean value theorem to obtain

$$\frac{|\varphi_A^n I^{(n)}|}{|I^{(n)}|} = \varphi_A^{n\prime}(x)$$

for some $x \in \overline{I^{(n)}}$. Thus

$$\min_{x \in \overline{I^{(n)}}} \varphi_A^{n\prime}(x) \leqq \frac{|\varphi_A^n I^{(n)}|}{|I^{(n)}|} \leqq \max_{x \in \overline{I^{(n)}}} \varphi_A^{n\prime}(x). \tag{19}$$

Again applying the mean value theorem we have

$$\min_{x \in \overline{I^{(n)}}} \varphi_A^{n\prime}(x) \leqq \frac{|E|}{|I^{(n)} \cap \varphi_A^{-n} E|} \leqq \max_{x \in \overline{I^{(n)}}} \varphi_A^{n\prime}(x) \tag{20}$$

for any interval $E \subset I^{(0)}$. We get (20) for arbitrary subsets E of $I^{(0)}$ of positive measure by applying a standard approximation argument in measure theory. Dividing (19) by (20) and recalling that $|\varphi_A^n I^{(n)}| = 1$ we obtain (18).

LEMMA 4. *There exists a constant* $C > 0$ *such that if* $I^{(n)} \subset I^{(1)}$ *then*

$$|I^{(n)}| \leqq C(1/2)^{n-1} |I^{(-1)}| \qquad \textit{for all } n \geqq 1. \tag{21}$$

PROOF. The reader might be tempted to apply (12) successively to get (21) but he would end up with the wrong $I^{(1)}$. The correct $I^{(1)}$ appears in the expression

$$I^{(n)} = I^{(1)} \cap \varphi_A^{-1} I^{(n-1)}. \tag{22}$$

We can apply Lemma 3 restricted to the case $n = 1$. We take E to be the set $I^{(n-1)}$ that appears in (22) and we get

$$\frac{|I^{(1)} \cap \varphi_A^{-1} I^{(n-1)}|}{|I^{(1)}| \cdot |I^{(n-1)}|} \leqq M_1(I^{(1)}) \leqq M_1.$$

Thus

$$\frac{|I^{(n)}|}{|I^{(1)}|} \leqq I^{(n-1)} \cdot M_1 \leqq 2^{-n+1} \cdot M_1.$$

From Lemma 2

$$M_1 \leqq (1 + C'') < \infty;$$

and we can take $C \equiv 1 + C''$.

LEMMA 5. *There exist a constant $M > 0$ such that*

$$M_n \leqq M < \infty \tag{23}$$

for all $n \geqq 1$.

PROOF.

$$M_{n+1}(I^{(n+1)}) = \sup_{x,y \in I^{(n+1)}} \frac{\varphi_A^{(n+1)'}(x)}{\varphi_A^{(n+1)}(y)}.$$

From the chain rule

$$\leqq M_n(I^{(n)}) \sup_{x \cdot y \in I^{(n+1)}} \frac{\varphi'_A(x)}{\varphi'_A(y)}$$

where $I^{(n)} = \varphi_A I^{(n+1)}$. Thus by Lemma 2

$$M_{n+1}(I^{(n+1)}) \leqq M_n \sup_{x \cdot y \in I^{(n+1)}} \left(1 + C'' \frac{|x - y|}{|I|}\right)$$

where $I^{(n+1)} \subset I$. Thus

$$\leqq M_n\left(1 + C'' \frac{|I^{(n+1)}|}{|I|}\right),$$

for $I^{(n+1)} \subset I$; and consequently by Lemma 4

$$\leqq M_n(1 + C \cdot C'' \cdot 2^{-n})$$

for all $I^{(n+1)}$.

Therefore

$$M_{n+1} \leqq M_n(1 + C \cdot C'' \cdot 2^{-n})$$

$$= M_1 \prod_{m=2}^{n} (1 + CC''2^{-n}),$$

and by Lemma 2

$$\leqq (1 + C'') \prod_{m=2}^{n} (1 + C \cdot C'' \cdot 2^{-m}).$$

Since the infinite product converges, we have

$$M_n \leqq M \equiv (1 + C'') \prod_{m=2}^{\infty} (1 + C \cdot C'' \cdot 2^{-m}) < \infty.$$

At last let us consider the tail field $\bigcap_{n=1}^{\infty} \varphi_A^{-n}\mathbb{B}_A$. For an element $E \in \bigcap_{n=1}^{\infty} \varphi_A^{-n}\mathbb{B}_A$ there exists a set $E_n \in \mathbb{B}_A$ for each n such that $E = \varphi_A^{-n}E_n$. The invariant sets of φ_A are tail sets. So are A^+ and A^-; however neither of these are invariant ($\varphi_A^{-1}A^+ = A^-$ and $\varphi_A^{-1}A^- = A^+$). Once we have proved that A^+ and A^- are the only nontrivial tail sets the proof of the main theorem is concluded.

LEMMA 6. *If E is a non-trivial tail set then E equals either A^+ or A^- up to a set of measure zero.*

PROOF. Let E be a tail set such that $|E \cap I^{(0)}| > 0$. It suffices to prove $|E \cap I^{(0)}| = |I^{(0)}|$. For each n there exists $E_{2n} \in \mathbb{B}_A$ such that $E \cap I^{(0)} = \varphi_A^{-2n}(E_{2n} \cap I^{(0)})$. From Lemmas 3 and 5, though we only need the left-hand inequality of (18),

$$|I^{(2n)}| \cdot |E \cap I^{(0)}|/M \leqq |E \cap I^{(0)} \cap I^{(2n)}| \tag{24}$$

for $I^{(2n)} \subset I^{(0)}$. Using the fact that α is a generator a standard approximation argument in measure theory allows us to replace the intervals $I^{(2n)}$ in (24) by arbitrary subsets of $I^{(0)}$, in particular by $E^c \cap I^{(0)}$. Therefore $|E^c \cap I^{(0)}| = 0$.

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